2. Boolean Lattices (3 lectures)

2.1. Symmetric Chains

We need some definitions

A chain C in P is $\boxed{\text{saturated}}$ if $C=(x_1<x_2<\dots<x_k)$ and $x_i \lessdot x_{i+1}$ that is C is unrefinable.

A poset is $\boxed{\text{graded}}$ or $\boxed{\text{ranked}}$ if every maximal chain has the same size. In this case the rank function $rk: P \to \{0,1,\dots,h(P)-1\}$ is defined by $rk(x)=0$ $\forall x \in Min(P)$

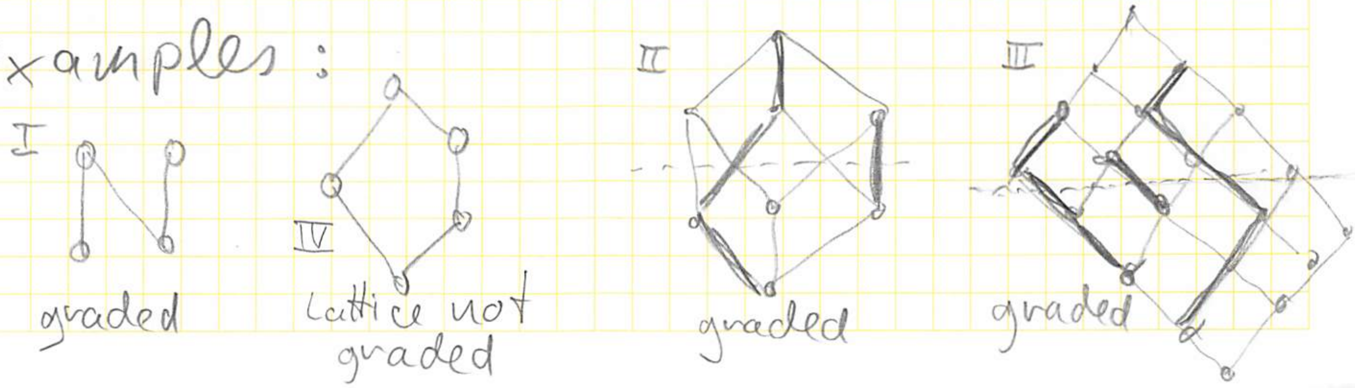
$rk(y)=r(x)+1$ if $x \lessdot y$

The rank of P is $rk(P)=h(P)-1$

A chain C in a ranked poset P is $\boxed{\text{symmetric}}$ if

- C is saturated
- $rk(min(C))+rk(max(C))=rk(P)$

Examples:



Symmetric chain decompositions for products of chains

A $\boxed{\text{sym chain decomp}}$ (SCD) is a decomposition (partition) of P into symmetric chains

Prop: if $\mathcal{C}$ is a SCD of P then $|\mathcal{C}| = w(P)$ and the middle rank $A = \{x : rk(x) = \lfloor \frac{1}{2} rk(P) \rfloor\}$ is a maximum antichain of P

Proof: Every chain $C \in \mathcal{C}$ contains an element of $A \Rightarrow |\mathcal{C}| \leq |A|$

$\Rightarrow |\mathcal{C}| = |A|$ whence $\mathcal{C}$ is minimum, A is maximum □

Even more is true: If P has a SCD and ρ_i = size of rank i and $r = rk(P)$

$\Rightarrow$ • $\rho_i = \rho_{r-i}$ • $\rho_{i-1} \leq \rho_i \quad \forall i \leq \lceil \frac{r}{2} \rceil$

if P has a SCD

Prop: The sequence $\rho_0\, \rho_1 \dots \rho_r$ of rank sizes is symmetric and unimodular.

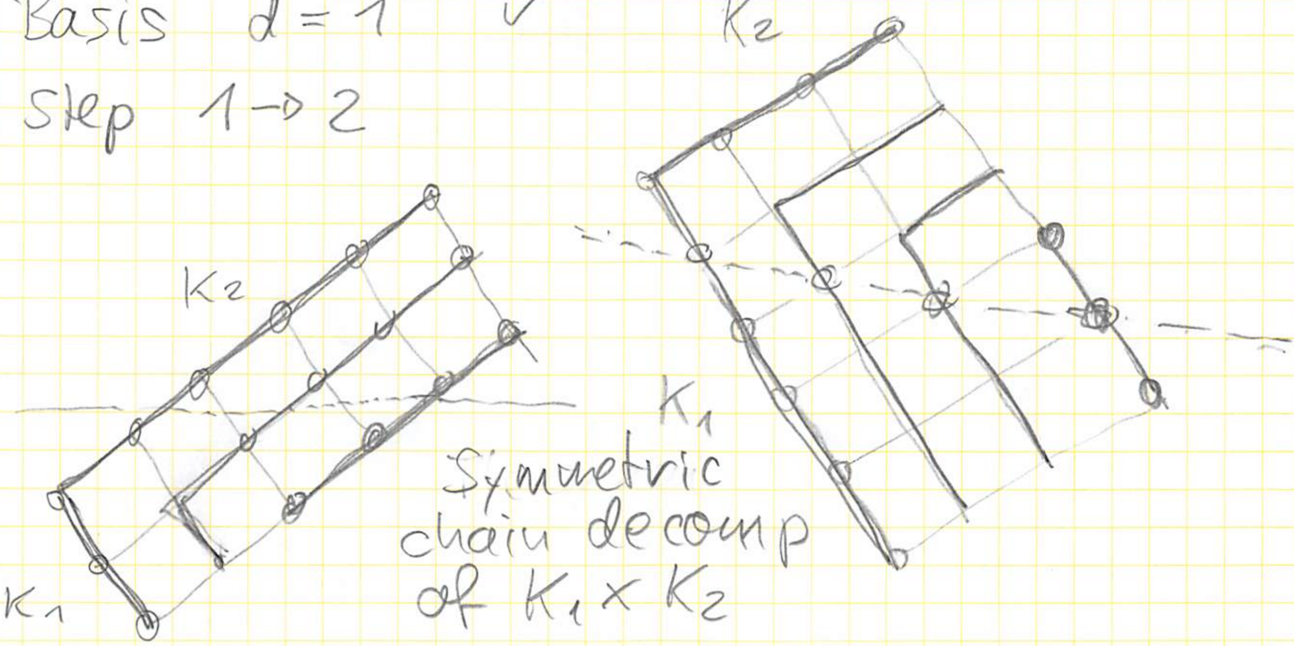
Theorem: Let $K_1 \dots K_d$ be chains

$\Rightarrow P = K_1 \times K_2 \times \dots \times K_d$ has a symmetric chain decomposition.

Proof: We use induction on d and let $rk(K_i) = k_i$

Basis $d = 1$ ✓

Step $1 \to 2$



Symmetric chain decomp of $K_1 \times K_2$

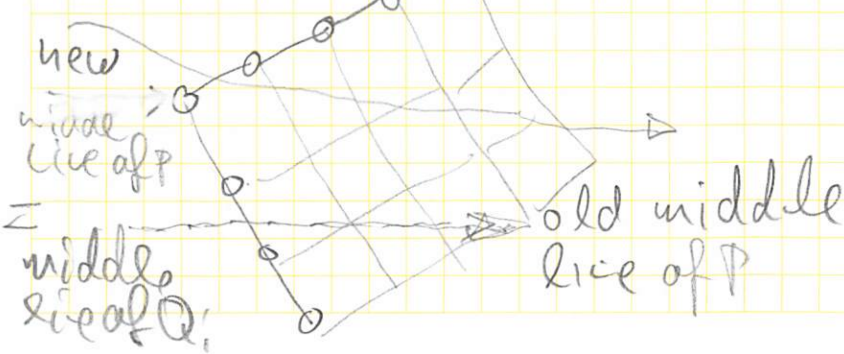
Step $d-1 \to d$

Let $\mathcal{C} = \{C_1, C_2 \dots C_w\}$ be a symm. chain decompositcon of $K_1 \times K_2 \times \dots \times K_{d-1}$.

Note that with $Q_i = C_i \times K_d$ we have a decomposition $Q_1 Q_2 \dots Q_d$ of P

Also $x \in Q_i$ and $rk_{Q_i}(x) = \left\lfloor \frac{1}{2} rk(Q_i) \right\rfloor$

$$\Rightarrow rk_P(x) = \left\lfloor \frac{1}{2} rk(P) \right\rfloor$$



The product with K_d raises the "middle line" of $K_1 \dots K_{d-1}$ as well as of C_i by $k_i/2$

$\Rightarrow$ If $\mathcal{C}_i$ is a symmetric chain decomp of Q_i $\forall i$ $\Rightarrow$ $\bigcup_{1}^{d} \mathcal{C}_i$ is a SCD of P $\square$

Corollaries:

- Divisor Lattices and Boolean Lattices have sym. chain decompositions

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- The width of $\mathcal{B}_n$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ (Sperner's Theorem)

A direct construction of a SCD of $\mathcal{B}_n$

Encodings of elements of $\mathcal{B}_n$ / subsets

$A = \{1, 3, 4, 7, 8\} \subseteq [10]$

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ ) & (&) &) & (& (&) &) & (& (\end{array}$$

matching parenthesis

matched elements $M_A = 2\,3\,5\,6\,7\,8$

free elements $F_A = 1, 4, 9, 10$

Let $A \subseteq [n]$, $F_A = (x_1 < x_2 < \dots < x_k)$

$I_A = A \cap M_A$, A sym. chain containing A:

$A_0 = I_A \quad A_1 = I_A + x_1 \quad \dots$

$\dots A_i = I_A + (x_1 \dots x_i) \quad \dots \quad A_k = I_A + F_A$

- saturated chain ✓

$|M_A| + |F_A| = rk(\mathcal{B}_n)$

$|A_0| = \frac{|M_A|}{2}$

$|A_k| = \frac{|M_A|}{2} + |F_A|$

Let P_A be the set of matched parenthesis of A

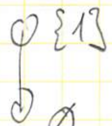
In our example $P_A = \{(23)(67)(58)\}$

Let $A \sim B$ if $P_A = P_B$ this is a equivalence relation. All the sets equiv to A are in the SC of A.

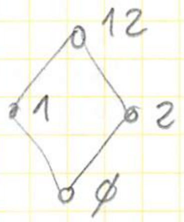
Proposition: The partition induced by $\sim$ on $\mathbb{B}_n$ is a SCD

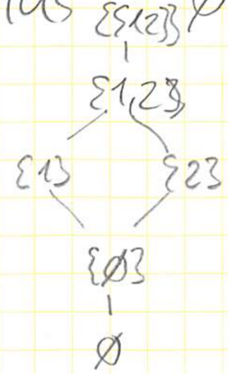
Application: The number of antichains of $\mathbb{B}_n$

Estimating the Dedekind numbers

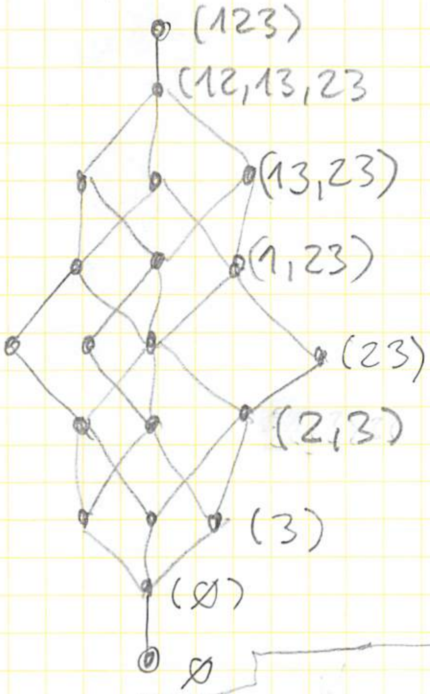
$\mathbb{B}_1$ 

Antichains $\emptyset$, $\{\emptyset\}$, $\{\{1\}\}$

$\mathbb{B}_2$ 

$\mathcal{F}_2$ 

$\mathcal{F}_n$ free distributive Lattice

$\mathcal{F}_3$ 

The sequence $M(n)$

$2, 3, 6, \overset{3}{20}, \overset{4}{168}, \overset{5}{7581}, \ldots$

$M(6) = 7828354$ ......

OEIS A000372

almost doubling of #digits

It is known that
$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \le \log_2 M(n) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + O\left(\frac{\log n}{n}\right)\right)$$

$\boxed{\text{Kleitman Markowsky '75}}$

Every subset of a middle rank of B_n is an antichain

$$\Rightarrow \quad M(n) \geq 2^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

Theorem: $M(n) \leq 3^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$

Proof: Antichains are in bijection to down sets, they are in bijection to monotone functions $f: 2^{[n]} \to \{0,1\}$

i.e to f with $A \subseteq B$, $f(B) = 1 \Rightarrow f(A) = 1$

With f we look at its restriction to each of the chains of a SCD

chain has size 1   2 possibilities

chain has size 2   3 possibilities

look at a chain of size $k+1$ after knowing f on all chains of size $\leq k$

C_A:

$$A_k = X_0 \supset X_1 \supset X_2 \supset \ldots \supset X_k$$
$$\vdots$$
$$A_2 = X_0 \supset X_1 \supset X_2 \subset \ldots \subset X_k$$
$$A_1 = X_0 \supset X_1 \subset X_2 \subset \ldots \subset X_k$$
$$A_0 = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_k$$

The X_i are blocks of $P(C_A) = P_{A_0}$ (sets of matched parentheses

For $i = 1 .. k-1$

$$A_i = \ldots\ldots) \, x_i \, (\ldots$$

Define $B_i = \text{-''-} \, (\, x_i \,) \, \text{-''-}$

B_i belongs to a shorter chain in the SCD

We know the value of $f(B_i)$

Also $A_{i-1} \subseteq B_i \subseteq A_{i+1}$

$\Rightarrow$ if $f(B_i) = 0 \Rightarrow f(A_j) = 0 \ \forall j < i$

if $f(B_i) = 1 \Rightarrow f(A_j) = 1 \ \forall j > i$

Look at the seq. $f(B_1) \, f(B_2) \ldots f(B_{k-1})$

if we see

$f(B)$:	0	1	0	1	(dont need this)
$f(A_i)$:		0	1		for C_A det

otherw: $f(B)$: $0 .. 0 \, 1 .. 1$ Two unknown values of f on C_A

$f(A)$ $.. 0 \, \square \, \square \, 1$

3 possibilities

All 0

$f(B)$ $0 \, 0$

$f(A)$ $0 \, \square \, \square$

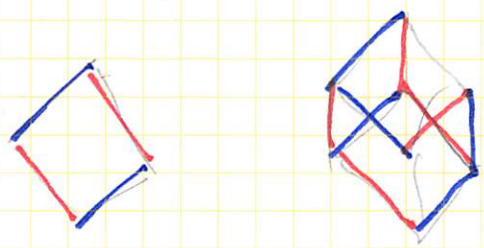
Orthogonal chain decompositions

Two chain partitions $\mathcal{C}_1, \mathcal{C}_2$ of P are orthogonal if $|C_1 \cap C_2| \leq 1 \;\forall C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2$

Proposition (Shearer Kleitman '79) For $n \geq 2$
B_n has a pair of orthogonal chain part. of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$

Conj $\lceil \frac{n+1}{2} \rceil$ pw orth. chain dec. exist

Proof:



2018: 4 pw orth for all $n \geq 60$ ind with comp. basis

Let $\mathcal{C}_1$ be the SCD obtained with
$) \leftrightarrow 1$, $(\leftrightarrow 0$ and matched pairs

Let $\mathcal{C}_2$ be obtained by the same construction with $) \leftrightarrow 0$, $(\leftrightarrow 1$

Claim If $C_1 \in \mathcal{C}_1$ $C_2 \in \mathcal{C}_2$ and $|C_1 \cap C_2| > 1$
then $C_1 \cap C_2 = \{\emptyset, [n]\}$

Suppose $Y, Z \in C_1 \cap C_2$ $Y < Z$

If Y not minimal in C_1

$Z = \quad\quad)X_i)$

$Y = X_0) \quad)X_i(\quad\quad (X_k$

in the C_2 encoding: $Y = \quad (\bar{X}_i)$

unmatched parentheses
* position unchanged in C_2

If z not maximal in C_1 (dual arg)

If y minimal and z maximal

In 2nd encoding one matched this pos matched $\emptyset$ unless $X_0 \dots X_{k-1}$ all empty

$z = X_0 \supset X_1 \supset \dots \supset X_k$

$y = X_0 \subset X_1 \subset \dots \subset X_k$

In the 2nd encoding $\boxed{\text{one of the } \subset}$ matched unless $X_1 \dots X_n$ all empty

$\Rightarrow y = \emptyset,\ z = [n]$

Move $\emptyset$ from its chain in $\mathcal{C}_2$ to another suitable chain. $\square$

Theorem: If P has a pair of orthogonal chain partitions $\mathcal{C}_1, \mathcal{C}_2$ with $|\mathcal{C}_1| = k$, $|\mathcal{C}_2| = l$ and elements x and y are chosen independently from some distribution on P

$$\Rightarrow \operatorname{Prob}(x \leq y) \geq \frac{1}{2}\left(\frac{1}{k} + \frac{1}{l}\right)$$

Remarks
- If $k = l = w(P)$
$$\Rightarrow \operatorname{Prob}(x \leq y) \geq 1/w(P)$$
- Applies to $\mathcal{B}_n$
- If C is an n-chain and we assume uniform distribution
$$\Rightarrow \operatorname{Prob}(x \leq y) = \frac{n+1}{2n} \sim \frac{1}{2} < \frac{1}{w(C)} = 1$$

Lemma 1: X set $|X| = n$ $p: X \to [0,1]$ prob-distribution, x and y independent

$$\Rightarrow \Pr(x = y) \geq 1/n$$

Rem: tight only if uniform distrib

Proof. $\Pr(X = Y) = \sum_{\alpha \in X} p_\alpha^2$

Now $0 \leq \sum_\alpha \left(p_\alpha - \frac{1}{n}\right)^2 = \sum_\alpha \left(p_\alpha^2 - \frac{2 p_\alpha}{n} + \frac{1}{n^2}\right)$

$$= \left(\sum_\alpha p_\alpha^2\right) - \frac{2}{n} + \frac{1}{n} \Rightarrow \Pr(x = y) \geq 1/n$$

$\square$

Lemma 2: $P = (X, \leq)$ partial order $p: X \to [0,1]$ prob-distr $\mathcal{C} = (C_1 \dots C_w)$ chain decomp, x and y independent

$$\Rightarrow \Pr(x, y \text{ belong to same } \mathcal{C} \text{ chain}) \geq 1/w$$

Proof: p induces $\hat{p}: \{C_1 \dots C_w\} \to [0,1]$

$$\Pr(x, y \text{ same chain}) = \hat{\Pr}(C_x = C_y) \underset{\text{Lem 1}}{\geq} 1/w \quad \square$$

Remark

$$\hat{\Pr}(C_x = C_y) = \sum_{\alpha \in X} p_\alpha^2 + 2 \sum_{\substack{\alpha < \beta \\ \alpha, \beta \text{ from same chain}}} p_\alpha p_\beta \geq 1/w$$

Proof of the theorem

$$\Pr(x \leq y) = \sum_{\alpha \in X} p_\alpha^2 + \sum_{\alpha < \beta} p_\alpha p_\beta$$

$$\geq \sum_{\alpha} p_\alpha^2 + \sum_{\substack{\alpha<\beta \\ \alpha\beta \text{ same } \mathcal{C}_1 \text{ chain}}} p_\alpha p_\beta + \sum_{\substack{\alpha<\beta \\ \alpha\beta \text{ same } \mathcal{C}_2 \text{ chain}}} p_\alpha p_\beta$$

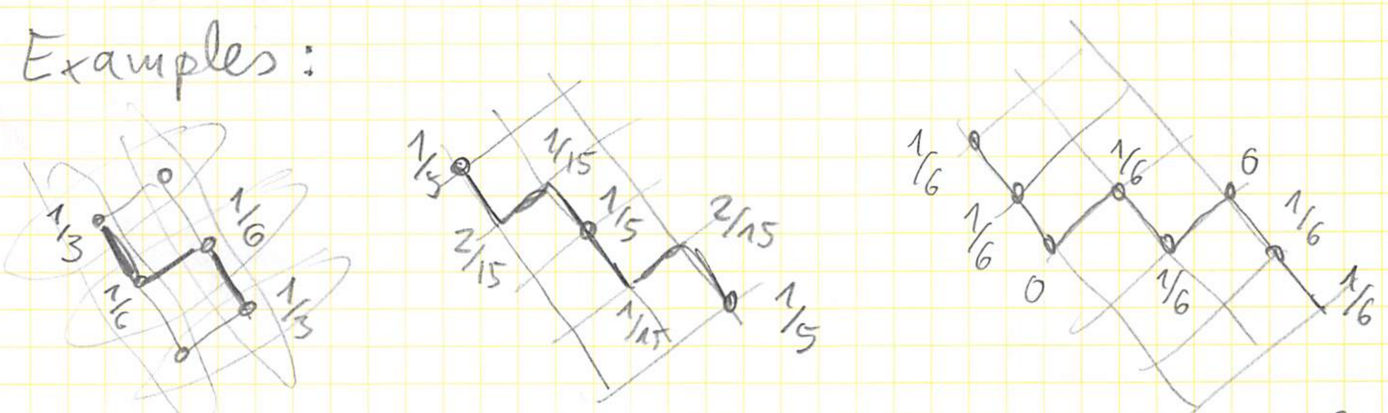
$$= \frac{1}{2}\Big(\sum_{\alpha} p_\alpha^2 + 2 \sum_{\substack{\alpha<\beta \\ \alpha\beta \text{ same } \mathcal{C}_1 \text{ chain}}} p_\alpha p_\beta\Big) + \frac{1}{2}\Big(\sum_{\alpha} p_\alpha^2 + \sum_{\substack{\alpha<\beta \\ \alpha\beta \text{ same } \mathcal{C}_2 \text{ chain}}} p_\alpha p_\beta\Big)$$

$$= \frac{1}{2}\Big(\hat{\Pr}_{\mathcal{C}_1}(C_x = C_y) + \hat{\Pr}_{\mathcal{C}_2}(C_x = C_y)\Big)$$

$$\geq \frac{1}{2}\Big(\frac{1}{k} + \frac{1}{\ell}\Big) \quad \square$$

Best possible consider $C_k \times C_\ell$

Examples:

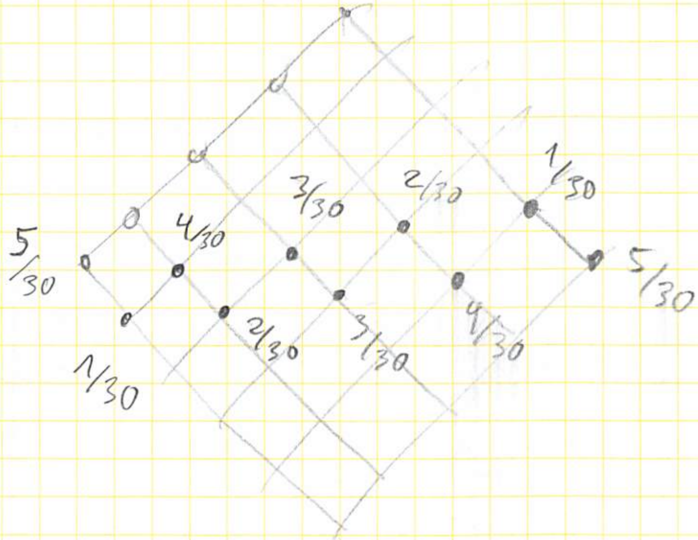


wenden

Each $C_k^\perp$ chain weight $1/k$

Each $C_\ell^\perp$ chain weight $1/\ell$

each comp pair $x \leq y$ both pos. prob in one of the chains


5/30
4/30
3/30
2/30
1/30
5/30
1/30
2/30
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A second proof of EKR

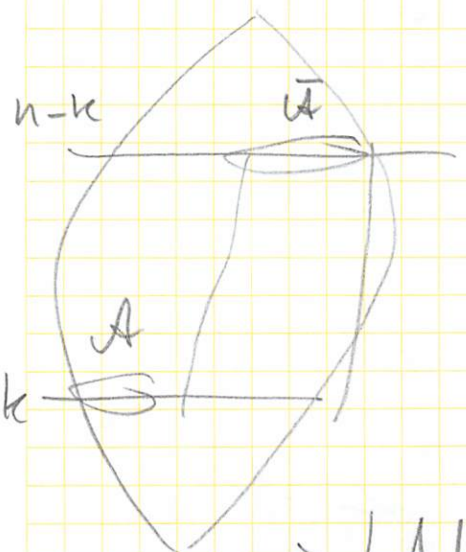
Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting $(n \geq 2k)$

Let $\bar{\mathcal{A}}$ be the family of complements

and note that $\mathcal{A} \cap D(\bar{\mathcal{A}}) = \emptyset$

or equivalently $\mathcal{A} \cap \underbrace{\Delta^{n-2k}(\bar{\mathcal{A}})} = \emptyset$

this is the shadow of $\bar{\mathcal{A}}$ on rank k



Suppose $|\mathcal{A}| > \binom{n-1}{k-1}$

$$\Rightarrow |\bar{\mathcal{A}}| = |\mathcal{A}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$$

$$\Rightarrow |\mathcal{A}| + |\Delta^{n-2k}\bar{\mathcal{A}}| > \binom{n-1}{k-1} + \binom{n-1}{(n-k)-(n-2k)}$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

$\lightning$

Übung $\mathcal{F}$ intersecting
$\Rightarrow S_i(\mathcal{F})$ intersecting

Frankl Buch Prop II.5.6

$S \subseteq \binom{[n]}{k}$ fam of arcs of circ. perm π

$\Delta_\pi(S)$ fam of $k-1$ arcs
show that unless $|S| = n$
$|\Delta_\pi(S)| > |S|$

Anders. Lemma 5.1.5

$kh \leq (h-1)n$

α cyclic permutation of $1 \dots n$
A_i k arcs such that
$A_{i_1} \cap A_{i_2} \dots \cap A_{i_h} \neq \emptyset$
$\Rightarrow$ at most k sets A_i

implies $\Rightarrow$ $\boxed{\text{Thm } 5.1.4}$

2.2 Shadows and intersecting families

Def: shadows

$\mathcal{B} \subseteq \binom{[n]}{k}$ k-uniform set family
a subset of the kth rank of B_n

$\Delta\mathcal{B} = \{A : |A| = k-1,\ \exists B \in \mathcal{B} \text{ with } A \subseteq B\}$

$\nabla\mathcal{B} = \{A : |A| = k+1,\ \exists B \in \mathcal{B} \text{ with } B \subseteq A\}$

Lemma (1) $|\Delta\mathcal{B}| \geq \frac{k}{n-k+1}|\mathcal{B}|$ (2) $|\nabla\mathcal{B}| \geq \frac{n-k}{k+1}|\mathcal{B}|$

Pr (1) Double count pairs (A,B)
$A \in \Delta\mathcal{B}$ $B \in \mathcal{B}$

(1) $(n-k+1)|\Delta\mathcal{B}| \geq \#(A,B) = k|\mathcal{B}|$

(2) $(k+1)|\nabla\mathcal{B}| \geq \#(A,B) = (n-k)|\mathcal{B}|$

□

Cor $|\Delta\mathcal{B}| \geq |\mathcal{B}|$ for $k \geq \frac{n+1}{2}$
$|\nabla\mathcal{B}| \geq |\mathcal{B}|$ for $k \leq \frac{n-1}{2}$

Sperner's proof of his theorem | LYM sym chain

$\mathcal{A}$ antichain
Let i be minimal such that
$\mathcal{A}_i := \mathcal{A} \cap \binom{[n]}{i} \neq \emptyset$

and suppose that $i < \frac{n-1}{2}$ Their definition

Let $\mathcal{A}' = \mathcal{A} - \mathcal{A}_i + \nabla \mathcal{A}_i$ it is an antichain

$|\mathcal{A}'| \geq |\mathcal{A}|$

With the dual construction we can make sure that

for all $i \geqslant \frac{n+1}{2}$ $\quad \mathcal{A} \cap \binom{[n]}{i} = \emptyset$

$\Rightarrow \mathcal{A} \subseteq \binom{n}{\frac{n}{2}}$ $\square$

We will come back to shadows

Intersecting families and the Erdős Ko Rado Theorem

Def: $\mathcal{F} \subseteq 2^{[n]}$ is intersecting if

$F \cap F' \neq \emptyset \quad \forall F, F' \in \mathcal{F}$

Question 1: What is the maximum size of an intersecting family $\mathcal{F} \subseteq 2^{[n]}$

Upper bound: Not $A, \bar{A} \in \mathcal{F}$

$\Rightarrow |\mathcal{F}| \leq \frac{1}{2} 2^n = 2^{n-1} \quad \forall \mathcal{F}$ intersecting

Lower bound:

$\mathcal{F}_x = \{ F \subseteq 2^n : x \in F \}$ is intersecting of size 2^{n-1}

Also $\mathcal{F}^+ = \{ F : |F| > \frac{n}{2} \}$ is intersecting if n odd its size $= 2^{n-1}$

5

Übung: For each $1 \leq k \leq \frac{n}{2}$ find an intersecting family $\mathcal{F}_k$ of size 2^{n-1} in $\mathcal{B}_n$ such that the smallest set in $\mathcal{F}_k$ has size k

Question 2: Find maximum size intersecting families in $\binom{[n]}{k}$

▷ if $k > \frac{n}{2}$ we can take $\mathcal{F}_k = \binom{[n]}{k}$

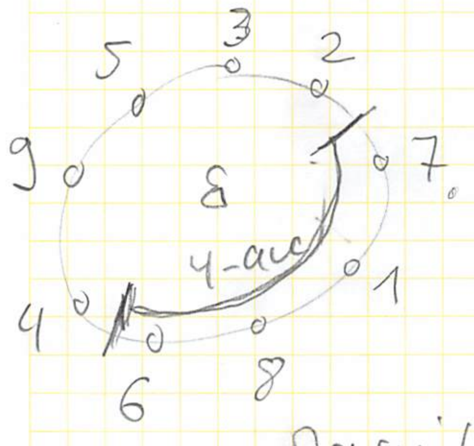
Thm [Erdös-Ko-Rado] discovered 1938 published 1961

if $2k \le n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ intersecting
then $|\mathcal{F}| \le \binom{n-1}{k-1}$

Remark Best possible
Consider $\mathcal{F}_x = \{ A \subseteq \binom{[n]}{k} : x \in A \}$
$\mathcal{F}_x$ is intersecting and $|\mathcal{F}| = \binom{n-1}{k-1}$.

We begin with a simple lemma

Consider a cyclic permutation σ of $[n]$



a k-arc on σ is an interval covering k elements of σ

Lemma: Let $2k \le n$ and let $\mathcal{A}$ be an intersecting family of k-arcs on $\sigma \Rightarrow |\mathcal{A}| \le k$

Proof Let $A \in \mathcal{A}$ every gap of A contains the endpoint of at most one k-arc of $\mathcal{A}$. There are $k-1$ gaps.

$\Rightarrow |\mathcal{A}| \le k$

Proof of the theorem

We double count pairs (A, π)

$A \in \mathcal{A}$, π cyclic perm A a k-arc of π

$$\sum_{\pi} \sum_{A \in \mathcal{A}} \delta[A \text{ arc of } \pi] \leq \sum_{\pi} k = (n-1)! \cdot k$$

$$\Vert$$

$$\sum_{A \in \mathcal{A}} \sum_{\pi} \delta[A \text{ arc of } \pi] = \sum_{A \in \mathcal{A}} k!\,(n-k)!$$

$$\Rightarrow |\mathcal{A}| \leq \frac{(n-1)!}{(k-1)!\,(n-k)!} = \binom{n-1}{k-1} \qquad \square$$

Shifting and the Kruskal-Katona Theorem

- shifting is a powerful combinatorial technique
- The KK-thm a strong tool with many application. It gives a precise lower bound on the size of shadows

We prepare for the theorem by defining the colex order on subsets of $\mathbb{N}$

$$A < B \iff \max(A \triangle B) \in B$$

If we take all subsets of $[4]$

1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
3	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
⋮	⋮															

compare with binary represent. of numbers

Relevant for us is the colex order on k subsets of $\mathbb{N}$

Example with $k=3$

					6				10								18			
1	1	1	0	1	1	0	1	0	0	1	1	0	1	0	0	1	0	0	0	1
1	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	1
1	0	1	1	0	1	1	0	0	1	0	1	1	0	0	1	0	0	1	0	0
0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	0
0				1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0
0										1	1	1	1	1	1	1	1	1	1	0
																				1

From the colex we derive the k-cascade number representation

Prop: $\forall k, m \quad \exists!\, a_k > a_{k-1} > \dots > a_s \geq s \geq 1$

such that $m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_s}{s}$

Example: representations for

$$10 = \binom{5}{3} \qquad 6 = \binom{4}{3} + \binom{2}{2} + \binom{1}{1}$$

$$18 = \binom{5}{3} + \binom{4}{2} + \binom{2}{1}$$

Existence: $a_k = \max\left(t : \binom{t}{k} \leq m\right)$

$$a_{k-i} = \max\left(t : \binom{t}{k-1} \leq m - \sum_{0}^{i-1} \binom{a_{k-j}}{k-j}\right)$$

Uniqueness: induction on k

Thm [Kruskal Katona]

$\mathcal{F} \subseteq \binom{\mathbb{N}}{k}$ $\quad |\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots \binom{a_s}{s}$

with $a_k > a_{k-1} > \dots > a_s \geq s \geq 1$

$\Rightarrow |\Delta \mathcal{F}| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \dots \binom{a_s}{s-1}$

Remark this is best possible, Look at initial segment of colex order on k sets

We show a slightly weaker but less technical theorem due to Lovász 1979

For $x \in \mathbb{R}$ let $\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$

$$\boxed{\text{Thm} \quad \mathcal{F} \subseteq \binom{\mathbb{N}}{k} \quad |\mathcal{F}| = \binom{x}{k} \text{ with } x \geq k \quad \Rightarrow |\Delta \mathcal{F}| \geq \binom{x}{k-1}}$$

Shifting: $\qquad$ the idea make the fam more similar to initial segment of colex

$\mathcal{F} \subseteq \binom{\mathbb{N}}{k}$ the ith shift operator $(i \geq 2)$ applied to $\mathcal{F}$ yields $S_i(\mathcal{F}) = \{S_i(F) : F \in \mathcal{F}\}$

where $S_i(F) = \begin{cases} F - i + 1 & \text{if } i \in F \quad 1 \notin F \\ & \quad F-i+1 \notin \mathcal{F} \\ F & \text{otherwise} \end{cases}$

Remark: More general shifting op. S_{ij} for $i<j$

$F \to F - j + i$

Observation $|S_i(\mathcal{F})| = |\mathcal{F}|$

If $S_i(F) = S_i(F') \Rightarrow S_i(F)$ not initially in $\mathcal{F}$
$\Rightarrow F = F'$

Lemma 1. $\Delta S_i(\mathcal{F}) \subseteq S_i \Delta(\mathcal{F})$

Proof: Let $E \in \Delta S_i(\mathcal{F})$, $E = S_i(F) - x$

We consider the intersection of $S_i(F)$ with $\{1, i\}$

- $\boxed{1, i \notin S_i(F)} \Rightarrow S_i(F) = F \Rightarrow E \subset F$
$\Rightarrow E \in \Delta F$ and $i \notin E \Rightarrow S_i(E) = E \in S_i(\Delta\mathcal{F})$
- $1, i \in S_i(F)$ because $i \in S_i(F)$ $F = S_i(F)$
$\Rightarrow E \in \Delta F$
if $x \neq 1 \Rightarrow 1 \in E \Rightarrow S_i(E) = E \Rightarrow E \in S_i(\Delta\mathcal{F})$
if $x = 1 \Rightarrow E' = E - i + 1 \in \Delta F$
$\Rightarrow E$ is blocked, $S_i(E) = E \Rightarrow E \in S_i(\Delta\mathcal{F})$
- $i \in S_i(F)$ $1 \notin S_i(F) \Rightarrow S_i(F) = F$
we note that F has been blocked whence
$F' = F - i + 1 \in \mathcal{F}$ $E \subset S_i(F) = F \Rightarrow E \in \Delta\mathcal{F}$
if $x = i \Rightarrow i \notin E$ so $E = S_i(E) \in S_i(\Delta\mathcal{F})$
if $x \neq i \Rightarrow E$ is being blocked by $E - i + 1 \in \Delta(F')$
$\Rightarrow E = S_i(E) \in S_i(\Delta\mathcal{F})$
- $i \notin S_i(F)$, $1 \in S_i(F) \Rightarrow i \notin E \Rightarrow S_i(E) = E$
If $E \in \Delta(\mathcal{F})$, then $E \in S_i(\Delta\mathcal{F})$
If F did not shift $S_i(F) = F$ and $E \in \Delta F$ ✓

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If F did shift $S_i(F) = F - i + 1$
if $x = 1$ then $E \subset F$ and $E \in \Delta \mathcal{F}$ ✓
if $x \neq 1$ Let $E' = E - 1 + i$ now $E' \subseteq F$
and $E' \in \Delta \mathcal{F}$ now S_i tries to
map E' to E hence $E \in S_i(\Delta \mathcal{F})$ □

A family $\mathcal{F}$ is stable if $S_i(\mathcal{F}) = \mathcal{F}$ $\forall i \geq 2$

Lemma 2 $\mathcal{F} \subseteq \binom{\mathbb{N}}{k}$ finite $\Rightarrow$ via shifting $\mathcal{F}$ can be converted into a stable fam. $\mathcal{G} \subseteq \binom{\mathbb{N}}{k}$ with $|\mathcal{F}| = |\mathcal{G}|$ and $|\Delta \mathcal{G}| \leq |\Delta \mathcal{F}|$

Proof: • Shifting preserves size (Obs)
• only decreases the shadow
• Each shift increases the number of sets containing 1 – terminates □

Given !!! a stable $\mathcal{F} \subseteq \binom{\mathbb{N}}{k}$ we partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$
where $\mathcal{F}_0 = \{F \in \mathcal{F} : 1 \notin F\}$ $\mathcal{F}_1 = \{F \in \mathcal{F} : 1 \in F\}$
we also Let $\mathcal{F}_1' = \{F - 1 : F \in \mathcal{F}_1\}$

Lemma 3: $|\mathcal{F}_1'| \geq |\Delta \mathcal{F}_0|$

proof: we show $\Delta \mathcal{F}_0 \subseteq \mathcal{F}_1'$.
Let $E \in \Delta \mathcal{F}_0$ $E = F - x$ for $F \in \mathcal{F}_0$, $x \neq 1$

$\mathcal{F}$ is stable $\Rightarrow$ $S_x(F) = F$ whence $F - x + 1 \in \mathcal{F}_1$

$\Rightarrow F - x \in \mathcal{F}_1'$ ie $E \in \mathcal{F}_1'$ $\square$

Lemma 4: $|\Delta \mathcal{F}| = |\mathcal{F}_1'| + |\Delta \mathcal{F}_1'|$

proof: By def $\Delta \mathcal{F} = \Delta \mathcal{F}_0 \cup \Delta \mathcal{F}_1$

by Lem 3 $\Delta \mathcal{F}_0 \subseteq \mathcal{F}_1'$ also by def $\mathcal{F}_1' \subseteq \Delta \mathcal{F}_1$

Hence $\Delta \mathcal{F} = \Delta \mathcal{F}_1$

Let $\mathcal{F}_1'' = \{ D + 1 ; D \in \Delta \mathcal{F}_1' \}$

we claim that $\Delta \mathcal{F}_1 = \mathcal{F}_1' \dot\cup \mathcal{F}_1''$

disjointness: $\mathcal{F}_1'$ no 1 | $\mathcal{F}_1''$ with 1

$\boxed{\text{"}\supseteq\text{"}}$

- $\mathcal{F}_1' = \{ F - 1 : F \in \mathcal{F}_1 \} \subseteq \Delta \mathcal{F}_1$
- $E \in \mathcal{F}_1'' \Rightarrow E = D + 1,\ D \in \Delta \mathcal{F}_1'$

  $\Rightarrow E \in \Delta \mathcal{F}_1$

$\boxed{\text{"}\subseteq\text{"}}$ $\Delta \mathcal{F}_1 \subseteq \mathcal{F}_1' \cup \mathcal{F}_1''$

$E = F - x \in \Delta \mathcal{F}_1$ if $x = 1$ $E \in \mathcal{F}_1'$

$x \neq 1 \Rightarrow F \in \mathcal{F}_1$

$$\begin{tikzcd}
 & F \in \mathcal{F}_1 \arrow[dl] \arrow[dr] & \\
F - 1 \arrow[dr] & & F - x \in \Delta \mathcal{F}_1,\ \in \mathcal{F}_1'' \\
 & F - 1 - x \in \Delta \mathcal{F}_1' \arrow[ur] &
\end{tikzcd}$$

$F - 1 \in \mathcal{F}_1''$

$\square$

Proof of Lovász Theorem

$\mathcal{F} \subseteq \binom{\mathbb{N}}{k}$ $|\mathcal{F}| = \binom{x}{k}$

$\Rightarrow |\Delta\mathcal{F}| \geq \binom{x}{k-1}$

induction on k

$k=1$: $|\mathcal{F}| = m = \binom{m}{1}$ $\Delta\mathcal{F} = \{\emptyset\}$ $|\Delta\mathcal{F}| = 1 = \binom{m}{0}$

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Now let $k \geq 2$

Inner induction on m

$m=1$ $m = 1 = \binom{k}{k}$ $|\Delta\mathcal{F}| = k = \binom{k}{k-1}$

Step $\mathcal{F} \subseteq \binom{\mathbb{N}}{k}$ $|\mathcal{F}| = m = \binom{x}{k}$

we may assume that $\mathcal{F}$ is stable (Lem 2)

by Lem 4 $|\Delta\mathcal{F}| = |\mathcal{F}_1'| + |\Delta\mathcal{F}_1'|$

Claim $|\mathcal{F}_1'| \geq \binom{x-1}{k-1}$

• Shagnik hat noch eine Fussnote für den Fall $x \in [k, k+1]$

The claim implies that $|\Delta\mathcal{F}| \geq \binom{x}{k-1}$

indeed $|\mathcal{F}_1'| \geq \binom{x-1}{k-1}$ and $|\Delta\mathcal{F}_1'| \underset{\text{induction}}{\geq} \binom{x-1}{k-2}$

$\binom{x-1}{k-1} + \binom{x-1}{k-2} = \binom{x}{k-1}$

Remark valid $\forall x \in \mathbb{N}_{\geq k-1}$

$\Rightarrow$ valid $x \in \mathbb{R}_{\geq k-1}$

prf of claim $|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1|$ and $|\mathcal{F}_1| = |\mathcal{F}_1'|$

Assume $|\mathcal{F}_1'| < \binom{x-1}{k-1}$

$\Rightarrow \binom{x}{k} = |\mathcal{F}_0| + |\mathcal{F}_1'| < |\mathcal{F}_0| + \binom{x-1}{k-1}$

$\Rightarrow |\mathcal{F}_0| > \binom{x}{k} - \binom{x-1}{k-1} = \binom{x-1}{k}$

Lemma 3 $|\mathcal{F}_1'| > |\Delta\mathcal{F}_0|$

$\Rightarrow |\mathcal{F}_1'| > |\Delta\mathcal{F}_0| \underset{\text{induction}}{\geq} \binom{x-1}{k-1}$ $\lightning$ $\square$

r - cross union families

chapt 14 acs
Frankl Tokushige

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Families $\mathcal{C}_1 \ldots \mathcal{C}_r \subseteq \binom{[n]}{k}$ are
r-cross union $\Leftrightarrow \forall (C_1 \ldots C_r)$ with $C_i \in \mathcal{C}_i$
$C_1 \cup C_2 \cup \ldots \cup C_r \neq [n]$

Remark: The complement families
$\overline{\mathcal{C}_i} = \{ \overline{C} : C \in \mathcal{C}_i \}$ are r-cross
intersecting, ie $\overline{C_1} \cap \ldots \cap \overline{C_r} \neq \emptyset$

Theorem [Frankl, Tokushige 2011]
Let $n \leq rk$ and $\mathcal{C}_1, \ldots, \mathcal{C}_r \subseteq \binom{[n]}{k}$ be
r-cross union $\Rightarrow \prod_1^r |\mathcal{C}_i| \leq \binom{n-1}{k}^r$

Remark: Let $r = 2$ and $\ell = n - k$
$\Rightarrow 2(k+\ell) = 2n$ whence $n \leq 2k \Leftrightarrow n \geq 2\ell$
Let $\mathcal{C}_1 = \mathcal{C}_2 = \overline{\mathcal{F}}$ with $\mathcal{F} \subseteq \binom{n}{\ell}$ inters.
The theorem implies $|\mathcal{F}| \leq \binom{n-1}{\ell-1}$ EKR

Proposition: Let $n \leq rk$ and $\mathcal{C}_1 \ldots \mathcal{C}_r \subseteq \binom{[n]}{k}$
be r-cross union Let $|\mathcal{C}_i| = \binom{x_i}{k}$
$\Rightarrow \sum_1^r x_i \leq r(n-1)$

Claim: The proposition implies the thm. 13

in the proof of the claim we use the Kruskal Katona thm

$$\prod_1^r |\mathcal{C}_i| = \binom{x_1}{k} \cdots \binom{x_r}{k} = \left(\frac{1}{k!}\right)^r \prod_{i=0}^{k-1} (x_1 - i) \cdots (x_r - i)$$

$$\leq \left(\frac{1}{k}\right)^r \prod_{i=0}^{k-1} \left(\frac{\sum x_i}{r} - i\right)^r$$

• Here we use the arithmetic-mean geometric-mean ineq. Cube has largest volume among all boxes with given sum of edgelength.

$$= \binom{\frac{1}{r}\sum x_i}{k}^r$$

use proposition

$$\leq \binom{n-1}{k}^r$$

Proof of prop. Induction on $s = rk - n$

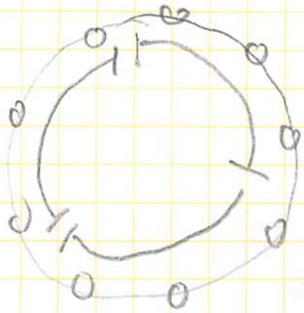
$\boxed{s=0}$ We use the circle method.

Let π be a cyclic permutation and let $\mathcal{C}_i(\pi)$ be the subfamily of sets of $\mathcal{C}_i$ which appear as arcs on π

$$\boxed{\text{Claim 1: } \sum_{i=1}^r |\mathcal{C}_i(\pi)| \leq r(n-k)}$$

proof: Let $\pi = (p_1 p_2 \cdots p_n($, $n = r \cdot k$

For $l = 1 \ldots n$ we define r successive disjoint k arcs $A_1^{(l)} \ldots A_r^{(l)}$ where $A_j^{(l)}$ has $p_{l+(j-1)k}$ as first element.



$n = 9$

$k = r = 3$

one of the families

Note that $\forall j$

$A_j^{(1)} \cdots A_j^{(n)}$

covers all leaves of π.

Hence $\sum_1^r |\mathcal{C}_i(\pi)| = \sum_{i=1}^r \sum_{\ell=1}^n \delta[A_i^{(\ell)} \in \mathcal{C}_i(\pi)]$

$$= \sum_{\ell=1}^n \#(j : A_j^{(\ell)} \in \mathcal{C}_j) \le n \cdot (r-1)$$

! note that for each ℓ there must be some j with $A_j^{(\ell)} \notin \mathcal{C}_i$

$$= rn - n = rn - rk = r(n-k)$$

□

Claim 2: If $n = rk$, then $\sum_1^r |\mathcal{C}_i| \le r\binom{n-1}{k}$

Proof

From Claim 1 we have

$$\sum_\pi \sum_i |\mathcal{C}_i(\pi)| \le (n-1)!\, r(n-k)$$

Each $C \in \mathcal{C}_i$ is an arc in $k!(n-k)!$ cyclic permutations

$$\sum_\pi |\mathcal{C}_i(\pi)| = |\mathcal{C}_i|\, k!(n-k)!$$

$$\Rightarrow \sum_1^r |\mathcal{C}_i| = \sum_{i=1}^r \frac{1}{k!(n-k)!} \sum_\pi |\mathcal{C}_i(\pi)|$$

$$= \frac{1}{k!(n-k)!} \sum_\pi \sum_i |\mathcal{C}_i(\pi)| \leq \frac{(n-1)!\, r\,(n-k)}{k\,(n-k)}$$

$$= r \binom{n-1}{k}$$

Since for $x \geq k$ $\binom{x}{k}$ is convex we get

$$\binom{\frac{1}{r}\sum_1^r x_i}{k} \leq \frac{1}{r}\sum_1^r \binom{x_i}{k} = \frac{1}{r}\sum_1^r |\mathcal{C}_i| \leq \binom{n-1}{k}$$

By monotonicity $\frac{1}{r}\sum x_i \leq n-1$

which yields $\sum x_i \leq r(n-1)$ for $s=0$

$\boxed{s \to s+1}$ so now $s+1 = rk - n$

Define $\mathcal{H}_i = \mathcal{C}_i \cup \mathcal{D}_i \subseteq \binom{[n+1]}{k}$

where $\mathcal{D}_i = \{ D + (n+1) : D \in \Delta \mathcal{C}_i \}$

From Kruskal Katona Lovasz version $|\mathcal{D}_i| \geq \binom{x_i}{k-1}$

$$|\mathcal{H}_i| \geq \binom{x_i}{k} + \binom{x_i}{k-1} = \binom{x_i+1}{k}$$

Let $|\mathcal{H}_i| = \binom{z_i}{k}$ and remember $z_i \geq x_i + 1$

Note that $\mathcal{H}_1 \dots \mathcal{H}_r$ is r-cross union 16

and $kr - (n+1) = s < s+1$

Now, by induction

$$\sum_1^r z_i \leq r((n+1)-1) = rn$$

$$\begin{array}{c} \vee\!\!| \\ \sum_1^r (x_i+1) = \sum x_i + r \end{array}$$

$$\Rightarrow \sum_1^r x_i \leq r(n-1) \qquad \square$$

2.3 Enumerating sets and binary vectors

Quelle: Wilf: Comb. Alg - an update

We have seen lists of sets, for example colex. In some applications it can be a disatvantage that going from the vector (11110) to the next (00001) all bits have to be changed

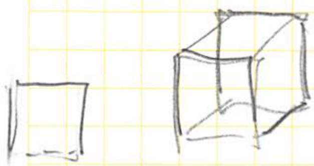
Frank Gray working at Bell Labs invented an enumeration to avoid this patent 1953

Gray code only makes a single bit flip when going from a vector to the next

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0

$G_n = (G_{n-1}\,0) \oplus (\overline{G_{n-1}}\,1)$

← read List backwards



The modern view to Gray Codes

$\mathcal{O}$ ← set of combinatorial objects

- k-element sets
- permutations
- triangulations of a polygon
- downsets of a poset
- linear extensions

ρ a flip operation which modifies elements of $\mathcal{O}$ locally

$(\mathcal{O}, \rho)$ defines a flip graph on $\mathcal{O}$

A Gray code is a Hamilton path (cycle) in the flip graph.

Problems addressed in this area

- existence
- construction algorithms
  - goal: ⇒ constant time per step
- ranking and unranking
  - find kth object of List
  - det position of object

Recommended
Knuth Combinatorial Algorithms

A recent highlight in the area of Gray Codes is Torsten Mütze's 2014 solution to the middle levels problem

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special case of a famous conj by Lovasz
hamiltonicity of connected vertex transitive graphs (some exceptions
(5) K_2/Petersen/
Coxeter/ P and C
by $\circ \to \triangle$

MLP raised in the 80s